Optimization of Information Storage With Quantum Walks

Gui Zhen Lu
CUNY Hunter College

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by

[Gui Zhen Lu]

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Thesis Sponsor:

August 9, 2019	[Mark Hillery]
Date	Signature
August 9, 2019	[János Bergou]
Date	Signature of Second Reader

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1 Overview

A four-vertex quantum graph was analyzed with the objective of storing the highest amplitude of an incoming qubit. The procedure included the use of phase shifters to allow the user to store and release information when he or she chooses. Several parameters, such as the phase shift, location of the phase shifter, the size and shape of the binding graph and initial incoming state were varied independently to optimize the storage capacity of the graph.

2 Introduction and Background to Quantum Walks

A quantum walk is a quantum version of a classical random walk. They were first proposed by Aharonov, Davidovich and Zagury [1]. They were re-introduced several years later after the advent of quantum information as a possible way to find new quantum algorithms [2]. Classical random walks serve as the basis of a number of classical algorithms, for example, those for graph search problems and satisfiability problems. The idea was that if one could produce a quantum version of a random walk, it could perhaps serve as a basis for new quantum algorithms. This led to a considerable research effort in the area of quantum walks, both theoretical and experimental, and a relatively recent review of the field can be found in [3].

The simplest classical random walk is a walk on a line. Time goes in steps, and the particle making the walk can move either one unit to the left or one unit to the right at each time step. A more colorful picture is that of a drunk taking steps at random, either to the left or to the right. For both the particle and the drunk, the direction of the next step is independent of the direction of the last one. What we would like to know is, after N steps, the probability of the particle being in a particular position.

Let us look at this in more detail. We will denote positions on the line by x, where x can be 0 or a positive or negative integer. The particle starts at $x = 0$, and at the next step will be at either $x = 1$ or $x = -1$. If the probability of moving left and moving right is the same, 1/2, then the probability of being at either of these locations after one step is 1/2. After two steps, the particle will be at either $x = 0$, with a probability of 1/2, or at either $x = 2$ or $x = -2$, each with a probability of 1/4. This comes about because the probability of any particular path of two steps is 1/4, and there is one path to get to $x = 2$, one to get to $x = -2$, and two paths to get from $x = 0$ back to $x = 0$. For N steps, the particle will have to be on an odd value of x, if N is odd, and an even value, if N is even. Consider the case N even. If the particle takes n_l steps to the left and n_r steps to the right, where $n_l + n_r = N$, then it will finish the walk at $x = n_r - n_l$. Note that these conditions imply that $n_r = (N + x)/2$ and $n_l = (N - x)/2$. Now each path has N steps, and the n_r right steps can be distributed among the total steps in $\begin{pmatrix} N \\ n_r \end{pmatrix}$ ways. Each path has a probability of $1/2^N$, so the probability of being at position x, for x even, after N steps (remember we are assuming N is even) is

$$p(x) = \frac{1}{2^N}\begin{pmatrix} N \\ n_r \end{pmatrix} = \frac{1}{2^N}\begin{pmatrix} N \\ (N+x)/2 \end{pmatrix}, \tag{1}$$

where in the last step, we have expressed n_r in terms of N and x.

This expression allows us to conclude two things about a random walk. The first is that the most likely place for the particle to be after N steps is where it started, at $x = 0$. This follows from the fact that the maximum of $\begin{pmatrix} N \\ n_r \end{pmatrix}$ occurs when $n_r = N/2$. The second is that the spread of the particle's position, $\Delta x = (\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$, where the angled brackets denote expectation value, is proportional to $\sqrt{N}$. This kind of spreading is characteristic of a diffusion process, so a quantum walk can be used to model such a process.

There are two kinds of discrete-time quantum walks. We will illustrate them by describing the walk on a line. Consider the line as made up of vertices at the locations of the integers connected by edges between adjacent vertices, e.g. the vertex at $x = 0$ is connected by an edge to the vertex at $x = 1$ and by another edge to the vertex at $x = -1$. This is an example of a graph, which is just a collection of vertices, some of which are connected by edges. Note that in the classical random walk, the particle sits on the vertices. This is also true in one of the versions of the quantum walk, the coined quantum walk [1]. For that walk, a two-state auxiliary system, the quantum coin, is required to make the dynamics unitary. The second kind of discrete-time quantum walk is called the scattering walk [5]. For that walk, the particle sits on the edges and no coin is necessary. We will be using the scattering walk throughout this thesis, and we will present a more detailed discussion of it shortly. For a walk on a line, the two kinds of quantum walk are isomorphic, but the scattering walk has a flexibility that makes it easier to use for more general graphs.

The coined quantum walk on a line was analyzed in [4]. Crucial differences between a classical random walk and the coined quantum walk were found. First, it is no longer the case that the particle is most likely to be where it started. Second, the spread in position, Δx, is found to be proportional to N, not $\sqrt{N}$. What accounts for the difference in the behavior is interference, which occurs in a quantum walk but not in a classical random walk. A quantum walk, rather than being related to diffusion, is closely related to wave propagation.

In graphs more complicated than a line there can exist localized states, which we call bound states. These are states that stay in one part of the graph no matter how many steps are taken. In this thesis, we would like to show how these states can be used to store information.

3 Scattering Walk

The graph used for a scattering quantum walk has vertices that are connected to each other by edges, just like the coined walk. But the particle sits on the edges and gets scattered at the vertices. Each edge has two states. The states specify the direction the particle is moving in. We can use the previous walk on a line as an example. We denote the state the particle is in by stating the vertices it is between, with the vertex it will interact with in the next time step second. For example, the state, $|-1,0\rangle$, tells us that the particle sits on the edge between the vertex labeled -1 and 0, facing the 0 vertex. After one discrete time step the particle will interact with the 0 vertex. The other state at the same location is $|0,-1\rangle$, which describes the particle facing the -1 vertex and would interact with that vertex in the next time step.

3.1 Transition Rules

There are two types of interactions a particle can have with vertices on a scattering walk. The particle can either transmit through the vertices or get reflected at the vertices. The transition rules for a vertex connected by two edges were found in [5] for a particle in the state $|j+1,j\rangle$ and $|j-1,j\rangle$ are respectively,

$$U|j+1,j\rangle = t^*|j,j-1\rangle - r^*|j,j+1\rangle) \tag{2}$$

$$U|j-1,j\rangle \quad = \quad t|j,j+1\rangle + r|j,j-1\rangle) \tag{3}$$

where t is the transmittance coefficient, r is the reflection coefficient and unitary requires $|t|^2 + |r|^2 = 1$. This thesis will be treating all vertices that are only connected by two edges as free particle propagation. This assigns t to equal to 1 and r to equal to 0.

The transition rules are a bit different for vertices that are connected by more than two edges, given the extra states to be accounted for. It was determined in [6] that the transmittance and reflection coefficients for any vertex with at least three edges can be found by:

$$r = \frac{n-2}{n} \tag{4}$$

$$t = \frac{2}{n} \tag{5}$$

where n is the number of edges attached to the vertex. It follows from unitary and the requirement that all the edges act the same way.

4 Storing States in a Graph

We will start off with a square graph that had been previous analyzed by [6].

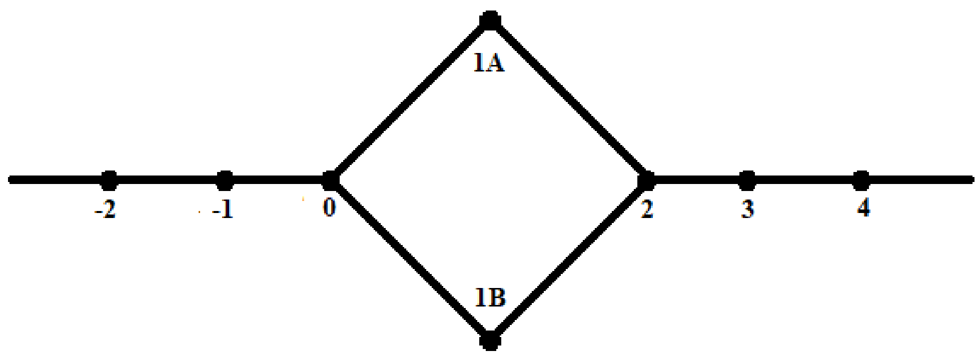

Figure 1: Square graph

On the graph above, the left side of the 0 vertex is connected to an infinitely long linear graph. The vertex 0 is attached to three edges, which are attached to vertices -1, 1A and 1B. Vertices 1A and 1B are also attached to vertex 2, which is connected to another infinitely long linear graph on the right side of the square graph. We can use Eqn. 4 and 5 to find the transmission and reflection coefficients at the 0 and 2 vertices. Since there are three edges attached each, we find that $t = \frac{2}{3}$ and $r = \frac{-1}{3}$ for both.

A free particle is set to be coming in from the left side, transmitting through every vertex until it hits the 0 vertex. Some of the particle will be reflected backwards, but most will be transmitted to the each of the two edges connected to 1A and 1B, entering the square graph. The particle will continue to act like a free particle until it hits vertex 2. A portion of the particle will be transmitted through vertex 2 and leak out of the square graph. The remainder will get transmitted and reflected within the square graph and travel towards the 0 vertex. The same thing will occur at the 0 vertex, when a portion of the particle will leak out of the square graph, while the remainder will then travel back towards the 2 vertex. Once enough time steps are taken, all of the particle will have leaked out of the square graph. This is because the particle is in an unbound state within the square graph.

It was determined in [5] that the bound states inside the square graph are the following:

$$|u_1\rangle = \frac{1}{\sqrt{2}}(|0,1A\rangle - |0,1B\rangle) \tag{6}$$

$$|u_2\rangle = \frac{1}{\sqrt{2}}(|1A,2\rangle - |1B,2\rangle) \tag{7}$$

$$|u_3\rangle = \frac{1}{\sqrt{2}}(|1B,0\rangle - |1A,0\rangle) \tag{8}$$

$$|u_4\rangle = \frac{1}{\sqrt{2}}(|2,1B\rangle - |2,1A\rangle) \tag{9}$$

A particle in any of these states or combination of these states would not be able to leak out of the square graph. When the particle reaches the 2 vertex, the two transmissions that occur onto the edge between vertices 2 and 3 end up destructively interfering with one another. The same thing occurs at the 0 vertex. I will now show this in detail with the state $|u_1\rangle$.

First we apply one time step onto $|u_1\rangle$:

$$U|u_1\rangle = \frac{1}{\sqrt{2}}(|1A, 2\rangle - |1B, 2\rangle) \tag{10}$$

This changes the particles state from $|u_1\rangle$ into $|u_2\rangle$, since the particle purely transmits through the 1A and 1B vertex. Next, we apply another time step.

$$U^2|u_1\rangle = \frac{1}{\sqrt{2}}(\frac{2}{3}|2, 3\rangle + \frac{2}{3}|2, 1B\rangle - \frac{1}{3}|2, 1A\rangle - \frac{2}{3}|2, 3\rangle - \frac{2}{3}|2, 1A\rangle + \frac{1}{3}|2, 1B\rangle) \tag{11}$$

We see that terms with the state $|2, 3\rangle$, first and fourth term, canceled each other out. The other two states can be combined together and we get the state $|u_4\rangle$:

$$U^2|u_1\rangle = \frac{1}{\sqrt{2}}(|2, 1B\rangle - |2, 1A\rangle) \tag{12}$$

It is not a coincidence that each time step resulted in the particle being in a different bound state. This shows that a particle in a bound state will stay bounded within the square and a particle that is in an unbound state will not. Therefore, we will need to introduce a phase shifter into the graph to get a particle from an unbound state into a bound state.

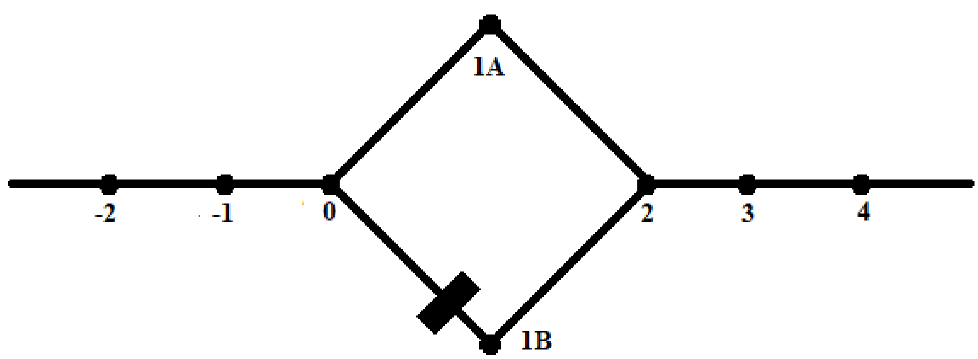

Figure 2: Square graph with phase shifter

A phase shifter will change the optical path difference of the particle by ϕ. A particle that passes through the phase shifter will pick up an $e^{i\phi}$ so that the state is still normalized. The figure above shows a phase shifter inserted on the left side of the 1B vertex. It is important to take note that a particle will not pick up a phase unless it passes through the left side of vertex 1B. For example, a particle going from a state of $|0, 1B\rangle$ to $|1B, 2\rangle$ will pick up a path difference of ϕ, but a particle going from the state $|1B, 2\rangle$ to $|2, 1A\rangle$ will not. Furthermore, if a particle was able to reflect off the 1B vertex, coming from the left, would pick up a phase difference of 2ϕ. However, if a particle was to reflect off of the 1B vertex from the right it would not pick up any phase difference. We will go into how we found the location of the phase shifter that optimizes the storage in a square graph later in the thesis, but we first must determine the appropriate ϕ value to transformation the unbound states into the bound states.

The first value we tried for ϕ was $\frac{\pi}{2}$. We can see the effects of this phase shift by analyzing a particle at the edge between the -1 and 0, coming from the left. We will call this state $|\psi\rangle$.

$$
\begin{aligned}
|\psi\rangle &= |-1, 0\rangle && (13)\\
U|\psi\rangle &= \frac{-1}{3}|0, -1\rangle + \frac{2}{3}|0, 1A\rangle + \frac{2}{3}|0, 1B\rangle && (14)
\end{aligned}
$$

After one time step, the most of the particle is inside the square, however it is still in an unbound state because it has not encountered the phase shifter. We will apply a few more time steps to find some resemblance of bound states.

$$
U^2|\psi\rangle = \frac{-1}{3}|-1, -2\rangle + \frac{2}{3}|1A, 2\rangle + \frac{2i}{3}|1B, 2\rangle \qquad (15)
$$

$$
\begin{aligned}
U^3|\psi\rangle &= \frac{-1}{3}|-2, -3\rangle + \frac{2}{3}(\frac{-1}{3}|2, 1A\rangle + \frac{2}{3}|2, 1B\rangle \\
&\quad + \frac{2}{3}|2, 3\rangle) + \frac{2i}{3}(\frac{-1}{3}|2, 1B\rangle + \frac{2}{3}|2, 3\rangle + \frac{2}{3}|2, 1A\rangle) \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
U^4|\psi\rangle &= \frac{-1}{3}|-3,-4\rangle + \frac{-2}{9}|1A,0\rangle + \frac{4e^{\frac{i\pi}{2}}}{9}|1B,0\rangle + \frac{4}{9}|3,4\rangle + \frac{2}{9}|1B,0\rangle \\
&\quad + \frac{4e^{\frac{i\pi}{2}}}{9}|3,4\rangle + \frac{4e^{\frac{i\pi}{2}}}{9}|1A,0\rangle
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
U^4|\psi\rangle &= \frac{-1}{3}|-3,-4\rangle + \frac{-2}{9}|1A,0\rangle + \frac{2}{9}|1B,0\rangle + \frac{4e^{\frac{i\pi}{2}}}{9}|1B,0\rangle + \frac{4e^{\frac{i\pi}{2}}}{9}|1A,0\rangle \\
&\quad + \frac{4}{9}|3,4\rangle + \frac{4e^{\frac{i\pi}{2}}}{9}|3,4\rangle
\end{aligned} \tag{18}
$$

After four time steps, we see that the second and third term makes up one of the bound states. However, we also see some of the particle is also leaking out to the right side of the square graph. In addition, the remaining non-bound states will also eventually leave the square graph, leaving only a small portion of the particle remaining inside. It turns out that by initially setting $\phi = \pi$, we can optimize how much of the particle will be bounded. We will use the same initial state to demonstrate this.

$$
|\psi\rangle = |-1,0\rangle \tag{19}
$$

$$
U|\psi\rangle = \frac{-1}{3}|0,-1\rangle + \frac{2}{3}|0,1A\rangle + \frac{2}{3}|0,1B\rangle \tag{20}
$$

$$
U^2|\psi\rangle = +\frac{2}{3}|1A,2\rangle - \frac{2}{3}|0,1B\rangle \tag{21}
$$

The first step is the same, regardless of the value of π. But after we apply another time step, we see that the particle is now in a bound state. Since the portion of the particle that got reflected at the 0 vertex will not be returning to the square well, the first term will be left out of the remainder of the analysis. Disregarding the reflection at the 0 vertex, the state now looks like $|u_2\rangle$ of the bound states. We can reset the phase shifter so that $\phi = 0$, as if to remove the phase shifter. This keeps the bound states within the square graph, as we have seen previously. We can leak the particle out of the square graph, some time steps down the line, by changing ϕ to π again, but we will analyze that later on in this thesis.

5 Storing a Qubit

Qubits are the building blocks of quantum information. A qubit can be represented by the superposition of two states, where the information is stored in the coefficients of the states. For our case, we will use a two state particle, with coefficients C_0 and C_1, and attempt to store it within the square graph. A particle is set to come in from the left with the following state, ψ.

$$|\psi\rangle = C_1|-2,-1\rangle + C_0|-1,0\rangle \tag{22}$$

We will use $\phi = \pi$ so that the qubit will be in the bound state after 3 steps.

$$U|\psi\rangle = C_1|-1,0\rangle - \frac{C_0}{3}|0,-1\rangle + \frac{2C_0}{3}|0,1A\rangle + \frac{2C_0}{3}|0,1B\rangle \tag{23}$$

$$\begin{aligned} U^2|\psi\rangle &= -\frac{C_1}{3}|0,-1\rangle + \frac{2C_1}{3}|0,1A\rangle + \frac{2C_1}{3}|0,1B\rangle - \frac{C_0}{3}|-1,-2\rangle \\ &\quad + \frac{2C_0}{3}|1A,2\rangle - \frac{2C_0}{3}|1B,2\rangle \end{aligned} \tag{24}$$

$$\begin{aligned} U^3|\psi\rangle &= -\frac{C_1}{3}|-1,-2\rangle + \frac{2C_1}{3}|1A,2\rangle - \frac{2C_1}{3}|1B,2\rangle - \frac{C_0}{3}|-2,-3\rangle \\ &\quad + \frac{4C_0}{9}|2,1B\rangle + \frac{4C_0}{9}|2,3\rangle - \frac{2C_0}{9}|2,1A\rangle + \frac{2C_0}{9}|2,1B\rangle \\ &\quad - \frac{4C_0}{9}|2,1A\rangle - \frac{4C_0}{9}|2,3\rangle \end{aligned} \tag{25}$$

$$U^3|\psi\rangle = -\frac{C_1}{3}|-1,-2\rangle + \frac{2C_1}{3}|1A,2\rangle - \frac{2C_1}{3}|1B,2\rangle - \frac{C_0}{3}|-2,-3\rangle + \frac{2C_0}{3}|2,1B\rangle - \frac{2C_0}{3}|2,1A\rangle \quad (26)$$

$$U^3|\psi\rangle = -\frac{C_1}{3}|-1,-2\rangle + \frac{2\sqrt{2}C_1}{3}|u_2\rangle - \frac{C_0}{3}|-2,-3\rangle + \frac{2\sqrt{2}C_0}{3}|u_4\rangle \quad (27)$$

After three steps, it appears we should reset the phase shifter to 0 and the entire qubit within the square graph will be bounded. If we reset the phase shifter too early the trailing portion of the particle would not have a chance to become bounded and if we reset the bound states too late the leading portion of the particle would start to leak out of the square graph through the 0 vertex. But it turns out we can actually store more of the qubit if we hold off on resetting the phase shifter. Lets take a few more steps to see how we can further maximize our storage.

$$U^4|\psi\rangle = -\frac{C_1}{3}|-2,-3\rangle + \frac{2C_1}{3}|2,1B\rangle - \frac{2C_1}{3}|2,1A\rangle - \frac{C_0}{3}|-3,-4\rangle - \frac{2C_0}{3}|1B,0\rangle - \frac{2C_0}{3}|1A,0\rangle \quad (28)$$

$$U^5|\psi\rangle = -\frac{C_1}{3}|-3,-4\rangle - \frac{2C_1}{3}|1B,0\rangle - \frac{2C_1}{3}|1A,0\rangle - \frac{C_0}{3}|-4,-5\rangle - \frac{2C_0}{3}(\frac{2}{3}|0,1A\rangle + \frac{2}{3}|0,-1\rangle - \frac{1}{3}|0,1B\rangle) - \frac{2C_0}{3}(\frac{2}{3}|0,1B\rangle + \frac{2}{3}|0,-1\rangle - \frac{1}{3}|0,1A\rangle \quad (29)$$

$$\begin{aligned} U^5|\psi\rangle &= -\frac{C_1}{3}|-3,-4\rangle - \frac{2C_1}{3}|1B,0\rangle - \frac{2C_1}{3}|1A,0\rangle - \frac{C_0}{3}|-4,-5\rangle \\ &-\frac{4C_0}{9}|0,1A\rangle - \frac{4C_0}{9}|0,-1\rangle + \frac{2C_0}{9}|0,1B\rangle - \frac{4C_0}{9}|0,1B\rangle \\ &-\frac{4C_0}{9}|0,-1\rangle + \frac{2C_0}{9}|0,1A\rangle \end{aligned} \tag{30}$$

$$\begin{aligned} U^5|\psi\rangle &= -\frac{C_1}{3}|-3,-4\rangle - \frac{2C_1}{3}(|1B,0\rangle + |1A,0\rangle) - \frac{C_0}{3}|-4,-5\rangle \\ &-\frac{2C_0}{9}(|0,1A\rangle + |0,1B\rangle) - \frac{8C_0}{9}|0,-1\rangle \end{aligned} \tag{31}$$

We see that a large portion of one state transmits through the 0 vertex after five time steps. However, we can actually use this to our advantage and attempt to increase the amplitude of the particle within the square graph. Since an amplitude of $-\frac{8C_0}{9}$ transmits out of the square graph, we can use destructive interference with the same qubit structure, but of different amplitude. To start building this qubit, we know that the its general form looks like:

$$D_1|k-1,k\rangle + D_0|k,k+1\rangle \tag{32}$$

After the fifth time step, we want the leading portion of this new qubit to match the state that would get transmitted out of the square graph of the initial qubit. This requires that $k=-1$ after the fourth time step. Since it behaves like a free particle, its initial value for k must be -5. Combining this requirement with the initial state ψ, we get a new state Ψ:

$$|\Psi\rangle = \frac{1}{N}(D_1|-6,-5\rangle + D_0|-5,-4\rangle + C_1|-2,-1\rangle + C_0|-1,0\rangle) \tag{33}$$

The factor of $\frac{1}{N}$ is to ensure that the state is still normalized. Therefore, N is related to amplitudes by

$$N^2 = C_0^2 + C_1^2 + D_0^2 + D_1^2 \tag{34}$$

To determine the relationship between the coefficients, we look at how this new particle looks after the fifth time step.

$$\begin{aligned} U^5|\Psi\rangle &= \frac{1}{N}(D_1|-1,0\rangle - \frac{D_0}{3}|0,-1\rangle + \frac{2D_0}{3}|0,1A\rangle + \frac{2D_0}{3}|0,1B\rangle \\ &\quad -\frac{C_1}{3}|-3,-4\rangle - \frac{2C_1}{3}(|1B,0\rangle + |1A,0\rangle) - \frac{C_0}{3}|-4,-5\rangle \\ &\quad -\frac{2C_0}{9}(|0,1A\rangle + |0,1B\rangle) - \frac{8C_0}{9}|0,-1\rangle) \end{aligned} \tag{35}$$

We can determine the value of D_0, relative to C_0.

$$-\frac{D_0}{3}|0,-1\rangle - \frac{8C_0}{9}|0,-1\rangle = 0 \tag{36}$$

$$D_0 = \frac{-8C_0}{3} \tag{37}$$

We can also use the same method to determine the relationship between D_1 and C_1 by take another time step. It turns out they are also proportional to each other by the same factor as D_0 to C_0. We substitute in these relationships and simply $|\Psi\rangle$ after the fifth time step.

$$\begin{aligned} U^5|\Psi\rangle &= \frac{1}{N}(-\frac{8C_1}{3}|-1,0\rangle - 2C_0|0,1A\rangle - 2C_0|0,1B\rangle - \frac{C_1}{3}|-3,-4\rangle \\ &\quad -\frac{2C_1}{3}(|1B,0\rangle + |1A,0\rangle) - \frac{C_0}{3}|-4,-5\rangle) \end{aligned} \tag{38}$$

Lets take two more time steps before resetting the ϕ back to 0.

$$U^6|\Psi\rangle = \frac{1}{N}(-2C_1|0,1A\rangle - 2C_1|0,1B\rangle - \frac{C_1}{3}|-4,-5\rangle - \frac{C_0}{3}|-5,-6\rangle - 2C_0|1A,2\rangle + 2C_0|1B,2\rangle) \quad (39)$$

$$U^7|\Psi\rangle = \frac{1}{N}(-\frac{C_1}{3}|-4,-5\rangle - \frac{C_0}{3}|-5,-6\rangle - 2C_1|1A,2\rangle + 2C_1|1B,2\rangle + 2C_0|2,1A\rangle - 2C_0|2,1B\rangle\frac{1}{N}) \quad (40)$$

$$U^7|\Psi\rangle = \frac{1}{N}(-\frac{C_1}{3}|-4,-5\rangle - \frac{C_0}{3}|-5,-6\rangle - 2\sqrt{2}C_1|u_2\rangle - 2\sqrt{2}C_0|u_4\rangle) \quad (41)$$

When we compare Eqn. 41 to Eqn. 27, we see that the particle is in the same bound states. Both equations are written in terms of C_0 and C_1. The coefficients of the portion of the state reflected at the 0 vertex differ in the two equations by a factor of $\frac{1}{N}$. However, the coefficients of the bound states differ in the two equations by a factor of $\frac{3}{N}$. Notice that we have tripled the relative amplitude of the bound states, and therefore increase the overall probability of the particle inside the bound region by nine-fold. As a matter of fact, repeating this method triples the relative amplitude of the bound states each time.

An important note to consider is that C_1 and C_0 are decoupled using this method. Because of the geometry and the symmetry of the square graph, the two consecutive incoming states would not interfere with each other at any point. The first state was always a step ahead of the second state. However, if one state had lagged behind by a certain amount of steps, it would be possible for the two states to interfere but not desirable for our case. It is necessary that no amplitude of a state depended on both C_1 and C_0, given that these coefficients are what holds the information. Wrapping up our findings together, we concluded that the ideal initial of the particle for maximum efficiency of storing information for this graph is:

$$
\begin{aligned}
|\Psi\rangle &= \frac{1}{N}[C_0(|-1,0\rangle - \sum_{n=1}^{M}\frac{8}{3}(-3)^{n-1}|-1-4n,0-4n\rangle) + C_1(|-2,-1\rangle \\
&\quad - \sum_{n=1}^{M}\frac{8}{3}(-3)^{n-1}|-2-4n,-1-4n\rangle)]
\end{aligned} \tag{42}
$$

We use N again here as a normalization factor where:

$$
N^2 = (C_0^2 + C_1^2)(1 + 64\sum_{n=1}^{M} 3^{2n-4}) \tag{43}
$$

The M term in Eq. 42 and 43 is the number of additional qubit states that we are redistributing the original state into. This number is also associated with the number of times we are using this destructive interference at the $|0,-1\rangle$ edge to keep more and more of the particle within the bounded graph. We should also take note increasing this M increases the normalization factor. As a result, this decreases the significance of the reflection at the 0 vertex when the particle first enters the square, if M is large and finite.

The factor of 4n in Eqn. 42 is due to the number of steps the particle takes before coming a full circle within the square well, similar to that of a period. This parameter would have to be adjusted to be compatible to the graph itself. Now that we determined the ideal state of the particle to store the most efficiency, we can turn our gear to the parameters of the graph itself to increase the number of qubits we can store.

5.1 Optimizing Storage Space with the Size of the Binding Graph

We can optimize the storage capacity of the graph by changing the size and shape of the bounded graph. It is necessary to increase the number of edges and vertices so that we would be able to store more qubits, and keep its symmetry to ensure the states of the qubits are decoupled. We will use a hexagon graph to demonstrate this.

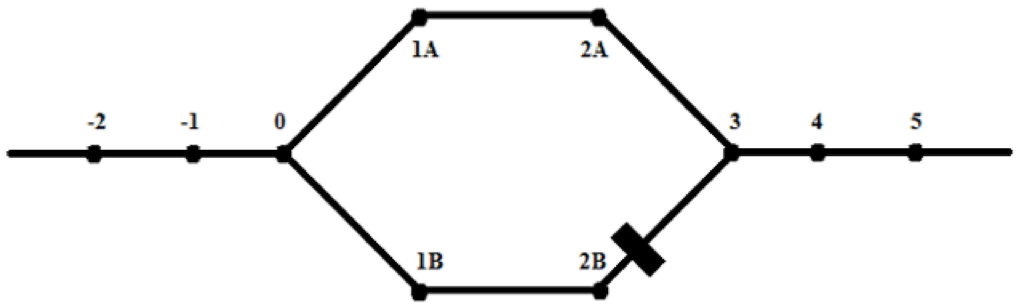

Figure 3: Quantum Walk Hexagon Graph with phase shifter attached to the right of the 2B vertex

The hexagon graph above is very similar to the square graph and the only difference is that there are two more vertices and edges within the binding graph. This graph is symmetric relative to the infinite line graphs attached to vertex 3 and vertex 0. The phase shifter is attached to the right of 2B, on the edge between 2B and 3. Since the hexagon graph has the same symmetry exhibited by the square graph, the bound states for this graph are:

$$|u_1\rangle = \frac{1}{\sqrt{2}}(|0,1A\rangle - |0,1B\rangle) \quad (44)$$

$$|u_2\rangle = \frac{1}{\sqrt{2}}(|1A,2A\rangle - |1B,2B\rangle) \quad (45)$$

$$|u_3\rangle = \frac{1}{\sqrt{2}}(|1B,0\rangle - |1A,0\rangle) \quad (46)$$

$$|u_4\rangle = \frac{1}{\sqrt{2}}(|2B,1B\rangle - |2A,1A\rangle) \quad (47)$$

$$|u_5\rangle = \frac{1}{\sqrt{2}}(|2B,3\rangle - |2A,3\rangle) \quad (48)$$

$$|u_6\rangle = \frac{1}{\sqrt{2}}(|3,2B\rangle - |3,2A\rangle) \quad (49)$$

To account for the size change of the graph, the unnormalized incoming state is composed of six consecutive states.

$$|\psi\rangle = C_5|-6,-5\rangle + C_4|-5,-4\rangle + C_3|-4,-3\rangle + C_2|-3,-2\rangle + C_1|-2,-1\rangle + C_0|-1,0\rangle \tag{50}$$

The state was advanced by six steps.

$$U|\psi\rangle = C_5|-5,-4\rangle + C_4|-4,-3\rangle + C_3|-3,-2\rangle + C_2|-2,-1\rangle + C_1|-1,0\rangle + \frac{2C_0}{3}|0,1A\rangle + \frac{2C_0}{3}|0,1B\rangle - \frac{C_0}{3}|0,-1\rangle \tag{51}$$

$$U^2|\psi\rangle = C_5|-4,-3\rangle + C_4|-3,-2\rangle + C_3|-2,-1\rangle + C_2|-1,0\rangle + \frac{2C_1}{3}|0,1A\rangle + \frac{2C_1}{3}|0,1B\rangle - \frac{C_1}{3}|0,-1\rangle + \frac{2C_0}{3}|1A,2A\rangle + \frac{2C_0}{3}|1B,2B\rangle - \frac{C_0}{3}|-1,-2\rangle \tag{52}$$

$$U^3|\psi\rangle = C_5|-3,-2\rangle + C_4|-2,-1\rangle + C_3|-1,0\rangle + \frac{2C_2}{3}|0,1A\rangle + \frac{2C_2}{3}|0,1B\rangle - \frac{C_2}{3}|0,-1\rangle + \frac{2C_1}{3}|1A,2A\rangle + \frac{2C_1}{3}|1B,2B\rangle - \frac{C_1}{3}|-1,-2\rangle + \frac{2C_0}{3}|2A,3\rangle - \frac{2C_0}{3}|2B,3\rangle - \frac{C_0}{3}|-2,-3\rangle \tag{53}$$

$$U^4|\psi\rangle = C_5|-2,-1\rangle + C_4|-1,0\rangle + \frac{2C_3}{3}|0,1A\rangle + \frac{2C_3}{3}|0,1B\rangle - \frac{C_3}{3}|0,-1\rangle + \frac{2C_2}{3}|1A,2A\rangle + \frac{2C_2}{3}|1B,2B\rangle - \frac{C_2}{3}|-1,-2\rangle$$

$$+\frac{2C_1}{3}|2A,3\rangle - \frac{2C_1}{3}|2B,3\rangle - \frac{C_1}{3}|-2,-3\rangle - \frac{2C_0}{3}|3,2A\rangle$$
$$+\frac{2C_0}{3}|3,2B\rangle - \frac{C_0}{3}|-3,-4\rangle \tag{54}$$

$$\begin{aligned} U^5|\psi\rangle &= C_5|-1,0\rangle + \frac{2C_4}{3}|0,1A\rangle + \frac{2C_4}{3}|0,1B\rangle - \frac{C_4}{3}|0,-1\rangle \\ &+\frac{2C_3}{3}|1A,2A\rangle + \frac{2C_3}{3}|1B,2B\rangle - \frac{C_3}{3}|-1,-2\rangle + \frac{2C_2}{3}|2A,3\rangle \\ &-\frac{2C_2}{3}|2B,3\rangle - \frac{C_2}{3}|-2,-3\rangle - \frac{2C_1}{3}|3,2A\rangle + \frac{2C_1}{3}|3,2B\rangle \\ &-\frac{C_1}{3}|-3,-4\rangle - \frac{2C_0}{3}|2B,1B\rangle - \frac{2C_0}{3}|2A,1A\rangle \\ &-\frac{C_0}{3}|-4,-5\rangle \end{aligned} \tag{55}$$

$$\begin{aligned} U^6|\psi\rangle &= \frac{2C_5}{3}|0,1A\rangle + \frac{2C_5}{3}|0,1B\rangle - \frac{C_5}{3}|0,-1\rangle + \frac{2C_4}{3}|1A,2A\rangle \\ &+\frac{2C_4}{3}|1B,2B\rangle - \frac{C_4}{3}|-1,-2\rangle + \frac{2C_3}{3}|2A,3\rangle - \frac{2C_3}{3}|2B,3\rangle \\ &-\frac{C_3}{3}|-2,-3\rangle - \frac{2C_2}{3}|3,2A\rangle + \frac{2C_2}{3}|3,2B\rangle - \frac{C_2}{3}|-3,-4\rangle \\ &-\frac{2C_1}{3}|2B,1B\rangle - \frac{2C_1}{3}|2A,1A\rangle - \frac{C_1}{3}|-4,-5\rangle - \frac{2C_0}{3}|1B,0\rangle \\ &-\frac{2C_0}{3}|1A,0\rangle - \frac{C_0}{3}|-5,-6\rangle \end{aligned} \tag{56}$$

The sixth step can be rewritten in terms of the bound states.

$$\begin{aligned} U^6|\psi\rangle &= \frac{2C_5}{3}|0,1A\rangle + \frac{2C_5}{3}|0,1B\rangle - \frac{C_5}{3}|0,-1\rangle + \frac{2C_4}{3}|1A,2A\rangle \\ &+\frac{2C_4}{3}|1B,2B\rangle - \frac{C_4}{3}|-1,-2\rangle - \frac{2\sqrt{2}C_3}{3}|u_5\rangle - \frac{C_3}{3}|-2,-3\rangle \\ &+\frac{2\sqrt{2}C_2}{3}|u_6\rangle - \frac{C_2}{3}|-3,-4\rangle - \frac{2C_1}{3}|2B,1B\rangle - \frac{2C_1}{3}|2A,1A\rangle \end{aligned}$$

$$-\frac{C_1}{3}|-4,-5\rangle - \frac{2C_0}{3}|1B,0\rangle - \frac{2C_0}{3}|1A,0\rangle - \frac{C_0}{3}|-5,-6\rangle \quad (57)$$

We can see after the advancing of these six steps that this graph can fit six decoupled states, or three qubits, within the bound states. However, only two of the six remained as bound states. It appears that only two states can remain bound states due to the location of the phase shifter. Figure 4 displays the optimal location for the phase shifter. By attaching the phase shifter to the bottom right of the 0 vertex, between 0 and 1B vertices, we can have up to six bound states within the hexagon graph at a time.

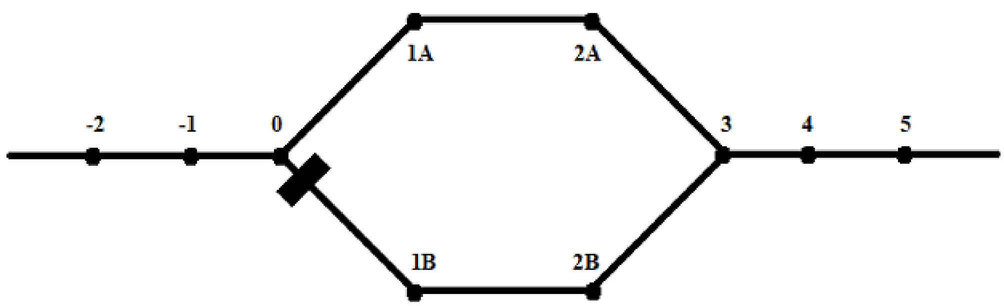

Figure 4: Quantum Walk Hexagon Graph with Phase Shifter at 0 Vertex

This is demonstrated by taking another six steps on the graph in Fig. 4 using the same initial state.

$$\begin{aligned} U|\psi\rangle &= C_5|-5,-4\rangle + C_4|-4,-3\rangle + C_3|-3,-2\rangle + C_2|-2,-1\rangle \\ &\quad + C_1|-1,0\rangle + \frac{2\sqrt{2}C_0}{3}|u_1\rangle - \frac{C_0}{3}|0,-1\rangle \end{aligned} \quad (58)$$

$$\begin{aligned} U^2|\psi\rangle &= C_5|-4,-3\rangle + C_4|-3,-2\rangle + C_3|-2,-1\rangle + C_2|-1,0\rangle \\ &\quad + \frac{2\sqrt{2}C_1}{3}|u_1\rangle + \frac{2\sqrt{2}C_0}{3}|u_2\rangle - \frac{C_1}{3}|0,-1\rangle - \frac{C_0}{3}|-1,-2\rangle \end{aligned} \quad (59)$$

$$
\begin{aligned}
U^3|\psi\rangle &= C_5|-3,-2\rangle + C_4|-2,-1\rangle + C_3|-1,0\rangle + \frac{2\sqrt{2}C_2}{3}|u_1\rangle \\
&\quad + \frac{2\sqrt{2}C_1}{3}|u_2\rangle - \frac{2\sqrt{2}C_0}{3}|u_5\rangle - \frac{C_2}{3}|0,-1\rangle \\
&\quad - \frac{C_1}{3}|-1,-2\rangle - \frac{C_0}{3}|-2,-3\rangle
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
U^4|\psi\rangle &= C_5|-2,-1\rangle + C_4|-1,0\rangle + \frac{2\sqrt{2}C_3}{3}|u_1\rangle + \frac{2\sqrt{2}C_2}{3}|u_2\rangle \\
&\quad - \frac{2\sqrt{2}C_1}{3}|u_5\rangle + \frac{2\sqrt{2}C_0}{3}|u_6\rangle - \frac{C_3}{3}|0,-1\rangle - \frac{C_2}{3}|-1,-2\rangle \\
&\quad - \frac{C_1}{3}|-2,-3\rangle - \frac{C_0}{3}|-3,-4\rangle
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
U^5|\psi\rangle &= C_5|-1,0\rangle + \frac{2\sqrt{2}C_4}{3}|u_1\rangle + \frac{2\sqrt{2}C_3}{3}|u_2\rangle - \frac{2\sqrt{2}C_2}{3}|u_5\rangle + \frac{2\sqrt{2}C_1}{3}|u_6\rangle \\
&\quad - \frac{2\sqrt{2}C_0}{3}|u_4\rangle - \frac{C_4}{3}|0,-1\rangle - \frac{C_3}{3}|-1,-2\rangle - \frac{C_2}{3}|-2,-3\rangle \\
&\quad - \frac{C_1}{3}|-3,-4\rangle - \frac{C_0}{3}|-4,-5\rangle
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
U^6|\psi\rangle &= \frac{2\sqrt{2}C_5}{3}|u_1\rangle + \frac{2\sqrt{2}C_4}{3}|u_2\rangle - \frac{2\sqrt{2}C_3}{3}|u_5\rangle + \frac{2\sqrt{2}C_2}{3}|u_6\rangle \\
&\quad - \frac{2\sqrt{2}C_1}{3}|u_4\rangle + \frac{2\sqrt{2}C_0}{3}|u_3\rangle - \frac{C_5}{3}|0,-1\rangle - \frac{C_4}{3}|-1,-2\rangle \\
&\quad - \frac{C_3}{3}|-2,-3\rangle - \frac{C_2}{3}|-3,-4\rangle - \frac{C_1}{3}|-4,-5\rangle \\
&\quad - \frac{C_0}{3}|-5,-6\rangle
\end{aligned} \tag{63}
$$

It is now clear that we can store 6 bound states within this graph. However, the same issue occurs, if we do not change the phase shift from π to 0 after the sixth step, in this hexagon graph as the square graph, after the fourth step. Some of the states will start to leak out from the 0 vertex. Fortunately, the symmetry of the square graph is preserved and we can solve this issue in a similar fashion as we did in the previous case. We can have additional states enter from the left infinite line graph to destructively interfere with the states that attempt to leave the bounded graph.

Since the solution was already found for the square graph, we can simply adjust the previous solution for this graph. We start with Eqn. 42 and change the $4n'$s to $6n'$s because that is how many steps it takes for a bound state to return the same bound state for the hexagon graph. In addition, we have to include additional terms for each of the states.

$$\begin{aligned}|\Psi\rangle &= \frac{1}{N}[C_0(|-1,0\rangle - \sum_{n=1}^{M}\frac{8}{3}(-3)^{n-1}|-1-6n,0-6n\rangle) \\ &+C_1(|-2,-1\rangle - \sum_{n=1}^{M}\frac{8}{3}(-3)^{n-1}|-2-6n,-1-6n\rangle) \\ &+C_2(|-3,-2\rangle - \sum_{n=1}^{M}\frac{8}{3}(-3)^{n-1}|-1-6n,0-6n\rangle) \\ &+C_3(|-4,-3\rangle - \sum_{n=1}^{M}\frac{8}{3}(-3)^{n-1}|-1-6n,0-6n\rangle) \\ &+C_4(|-5,-4\rangle - \sum_{n=1}^{M}\frac{8}{3}(-3)^{n-1}|-1-6n,0-6n\rangle) \\ &+C_5(|-6,-5\rangle - \sum_{n=1}^{M}\frac{8}{3}(-3)^{n-1}|-1-6n,0-6n\rangle)] \end{aligned} \tag{64}$$

Of course, the normalization factor do have to be adjusted for the additional quibits.

$$N^2 = (C_0^2 + C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2)(1 + 64\sum_{n=1}^{M} 3^{2n-4}) \tag{65}$$

6 Discussion

We have shown that we can optimize the storage capacity of a graph by vary numerous parameters of a quantum walk of a particle entering a bounded graph. Based on our findings, there are two parameters that restrict the relative amplitude of the states stored; the phase shift and the initial state of the particle. We determined that a phase shift of π can convert all of the unbound states that undergo the phase shift to become bound states. We also designed the optimal initial state of the particle to prevent the loss of the amplitude due to transmission at the 0 vertex out of the bounded graphs. Since the initial state of the particle is made of the superposition of an infinite number of duplicated individual states continuously coming into the bounded graph, each with higher amplitude, the amplitude of the states being reflected at the 0 vertex becomes negligible.

We also see that the number of stored qubits capacity is restricted by the placement of the phase shifter and the size of the bounded graph. The size of the bounded graph limits how many bounded states the graph can hold. Having a small bounded graph will only allow one to store a small number of qubits. However, the converse is not true. Although a graph can have a large number of bounded states, the number of stored states is then restricted by the location of the phase shifter. The closer the phase shifter is to the entrance of the bounded graph, the higher the storage capacity of the graph is.

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